Phys 410
Spring 2013
Lecture #25 Summary
27 March, 2013

We considered the two-body problem of two objects interacting by means of a conservative central force, with no other external forces acting. After transforming to the center of mass reference frame and exploiting conservation of angular momentum ℓ, the radial equation of motion can be written as $\mu\ddot{r} = \ell^2/\mu r^3 - dU/dr$. The first term on the RHS can be written in terms of a derivative as $-\frac{d}{dr}(\ell^2/2\mu r^2)$, so that it can be combined with the potential to create a new "effective potential" $U_{eff}(r) = U(r) + \ell^2/2\mu r^2$. The equation of motion finally reduces to a simple one-dimensional form: $\mu\ddot{r} = -dU_{eff}/dr$.

The effective potential (for $\ell > 0$) has a minimum at a finite value of r, diverges as r goes to zero, and approaches zero from below as r goes to infinity. We found that mechanical energy for the relative coordinate is conserved and equal to $E = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} + U(r)$. Since kinetic energy is either positive or zero, the particle must be located in a region where $E \geq U_{eff,min}$. We see that when $E > 0$ the particle has an un-bounded orbit, while when $E < 0$ it has a bounded orbit trapped between minimum and maximum values of r.

We then solved the radial equation $\mu\ddot{r} = \frac{\ell^2}{\mu r^3} + F(r)$ for inverse-square force-laws of the form $F(r) = -\gamma/r^2$, and found a solution that expressed the radial coordinate in terms of the angular polar coordinate, $r = r(\varphi)$, in which time has been eliminated. The result is $r(\varphi) = \frac{c}{1+\epsilon\cos\varphi}$, where $c = \frac{\ell^2}{\mu\gamma}$ is a length scale and ϵ is an un-determined constant. This is the equation for the orbit of a planet around the sun, or a satellite around the earth.

Note that when the un-determined constant $\epsilon > 1$, the denominator of $r(\varphi)$ has a zero for some angle φ, and the particle is off at infinity for that angle. This is an un-bounded orbit, like those with energy $E > 0$ noted above. When $\epsilon < 1$ the values of $r(\varphi)$ are finite for all φ, and the orbit is bounded, like those with $E < 0$ noted above.